Solving Bicoloring-Graph Games on Rectangular Boards – Part 1: partisan Col and Snort

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Abstract. In this paper we give an overview of results obtained for solving the combinatorial games of Col and Snort on rectangular boards. For Col on boards with at least one dimension even we give a strategy guaranteeing a win for the second player. For Col on general odd × odd boards we found no applicable strategy, though all experimental data show second-player wins. For Linear Col we were able to prove using Combinatorial Game Theory (CGT) that all chains, including odd-length chains) are second-player wins.
A similar strategy as for Col guarantees for Snort on boards with both dimensions even a win for the second player and with at least one dimension odd a first-player win. Snort therefore is completely solved.

1 Introduction

In Artificial Intelligence map-coloring has been a prime focus of research. In its basic form the question is: can a map with neighboring regions be colored with some finite number of colors, such that neighboring regions are colored differently? Any map-coloring problem is equivalent with some graph-coloring problem, where nodes represent regions, and edges denote common frontiers between corresponding regions, and the goal is to color all nodes in the graph such that two connected nodes are colored differently.

In the field of Combinatorial Game Theory (CGT in short), graph-coloring problems can be transformed into games by changing the goal of a game: not to fully color a map, but to make the last move (under the normal ending rule) when players alternately color one region. It is common to restrict such graph-coloring games to two colors, where both players have their own color, conventionally Black for the player who starts the game and White for the opponent.

The two combinatorial graph-coloring games most well known are surely Col and Snort, both first analyzed by Conway [3]. He attributed Col to Colin Vout and Snort to Simon Norton. Both are similar in the sense that both players alternately color a node in the graph, where one player may only color it black, the other only white. The two games differ in their conditions for coloring: in Col neighboring nodes may not be colored the same (further called the Col-condition), while in Snort they may not be colored differently (the Snort-condition).

Although both games can be played on any types of graphs, in this paper we concentrate on rectangular boards (sometimes just refered to as *boards*), where both players alternately put a stone of their color on a square. As a special case of Col and Snort on boards we consider Linear Col and Snort, played on one-dimensional boards (further called *chains*).

The literature on Col and Snort is very scarce. It has been introduced in the framework of CGT in the seminal books *On Numbers and Games* [3] and *Winning Ways* [2], where many small graphs are given and some more general rules are exemplified. Most of these are irrelevant for analyzing larger boards, except Linear Col, for which values were given without proof. Such a proof is given in this paper. Recently, a bachelor thesis by Demeur [4] reports solving many Col and Snort boards with sizes up to some 30 squares, based on $\alpha\beta$ search. We are not aware of any further analyses of Col and Snort.

2 Combinatorial Game Theory for Col and Snort

In this section we give a short introduction to the Combinatorial Game Theory as far as relevant for Col and Snort. For a more thorough introduction, we refer to the literature, in particular [3, 2, 1, 6].

In a combinatorial game, the players are conventionally called Left and Right. For Col and Snort, Left is the player moving the black stones, therefore also denoted as Black, and similarly Right (White) moves the white stones. A game G is then represented as $G = \{G^L \mid G^R\}$, where G^L and G^R stand for sets of games (the *options*) that players Left and Right, respectively, can reach by making one move in the game. The *value* of a game indicates how good a game is for a player. Then there are four possible outcome classes.

1. The class $\mathcal{L}$ consists of all positions where Left wins, irrespective of who moves first. These positions have strictly positive values.
2. The class $\mathcal{R}$ consists of all positions where Right wins, irrespective of who moves first. These positions have strictly negative values.
3. The class $\mathcal{N}$ consists of all positions where the player to move (the next player) wins. These positions have fuzzy values (incomparable with 0).
4. The class $\mathcal{P}$ consists of all positions where the player to move loses, so the previous player wins. These positions all have value 0.

Depending on the outcome class of a game, several types of values are possible. We treat the most important ones for Col and Snort in the next subsections.

2.1 Numbers and Star

Numbers have the property that any option is a number itself, and that any left option has a lower value than any right option. The simplest number game is the endgame $\{|\}$, denoted as 0. In this position, no player has any available moves, so it is a loss for the player to move. Larger or smaller numbers are built recursively. So $0 = \{|\}; 1 = \{0\ |\}; 2 = \{1\ |\}; -1 = \{|\ 0\}; -2 = \{|\ -1\}$; etc.

Some example Col positions with integer values are given in Fig. 1. In the left position, there is only one empty square, but due to the Col-condition it can be colored neither black nor white, so this position has value 0. In the middle position there is also one empty square, which may only be colored black, so this position has value +1. In the right position, with two empty squares, only White can move, so this position has value −2.

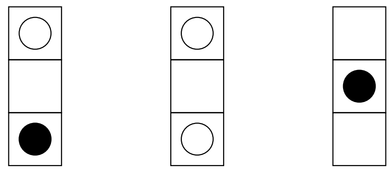

Fig. 1. Example Col positions on the 3×1 board with integer values.

Also fractions are possible. For example, the position in Fig. 2 has value $\{-1, 0 \mid 1\} = \{0 \mid 1\}$. Naturally this value is notated as $1/2$ (supported by the proof that two games with value $1/2$ are equivalent to one game with value 1).

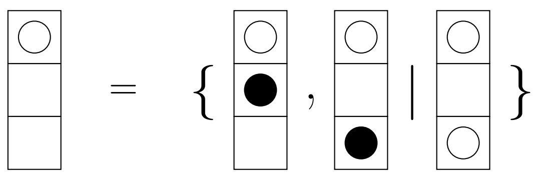

Fig. 2. Example Col position on the 3×1 board with value $1/2$.

Besides the endgame 0 on which all numbers are built, the most important simple game is the one denoted as Star or $*$. It is defined as $* = \{0 \mid 0\}$, where the player to move has just 1 option, leading to the endgame. Therefore, whereas the game 0 is a game where the second player to move wins (since trivially the next player cannot move), the $*$ is a game where the first player to move wins. A trivial example in both Col and Snort is a lone empty square.

$*$ is a fuzzy number, incomparable with 0. In fact it is a nimber, which is formally defined as $*n = \{*0, *1, *2, \cdots, *(n-1) \mid *0, *1, *2, \cdots, *(n-1)\}$. In partisan games like Col and Snort nimbers can occur, but are quite rare; so far we only found nimbers $0 = *0$ and $* = *1$. Conway [3] has proven that in Col every position has as value a number (z) or a number plus $*$ (notation $z*$).

2.2 Switches

Simple Snort games often have numbers as options, but have at least one left option with a larger value than some right option. For simple switches of the form $\{a \mid b\}$ ($a > b$) an alternate notation is $\frac{a+b}{2} \pm \frac{a-b}{2}$, where the first term is the *mean* value of the switch and the second term its *temperature*. A few example Snort positions with simple switches as values are given in Fig. 3.

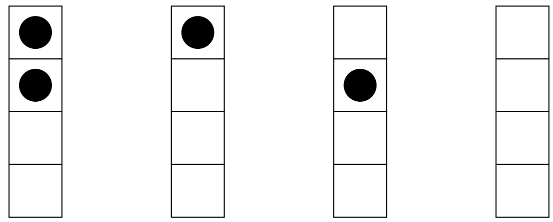

Fig. 3. Example Snort positions on the 4×1 board with switch values.

In the left position Black to move can take any of the remaining two empty squares to reach a position of value $+1$, whereas White's only option is to take the lowest empty square, ending the game; so this position is a switch with value $\{1 \mid 0\}$ (alternate notation $1/2 \pm 1/2$). The second and third positions likewise yield values $\{2 \mid -1\}$ (alternately $1/2 \pm 1 1/2$) and $\{2 \mid 1\}$ (alternately $1 1/2 \pm 1/2$). Clearly the third position is to be preferred for Black over the second one. As a consequence, in the rightmost position this option for Black dominates; since White options are the negation of Black's options, the latter position can be seen to have value $\{\{2 \mid 1\} \mid \{-1 \mid -2\}\}$ (alternately $\pm\{2 \mid 1\}$). It is clear that larger boards can have quite long and complicated switches as values.

Note that switches of the form $\pm x$ are called *fair switches*, since both players to move gain the same profit. Obviously, all switches for empty Snort boards are fair switches.

3 Partisan Col on Rectangular Boards

In this section, we investigate standard (*partisan*) Col on rectangular boards. This means that both players have their own stones, black for the Left player and white for the Right player.

3.1 Col on $m \times n$ boards with m and/or n even

For $m \times n$ Col boards with m and/or n even we found that the second player always can win. This can easily be proven as stated in Theorem 1.

Theorem 1. *All empty $m \times n$ Col boards with m and/or n even are second-player wins and thus have CGT value 0.*

Proof. The second player can use a copy-strategy as follows. Whereever the first player (Black) moves, the second player (White) plays symmetric wrt the centre. Then after every black move the board has opposite-color symmetry wrt to centre. Since every black move must fulfil the Col-condition, every white move will automatically fulfil the Col-condition also. Consequently, the second player makes the last move and wins. □

We further denote this strategy as the *centre strategy*. Example Col games on the 4×4 and 4×5 boards where White uses this winning strategy are shown in Fig. 4. The left diagram shows a Col game on an even $\times$ even board, the right diagram on an even $\times$ odd board. The numbers inside the stones are the move numbers. The small dot indicates the centre of the board.

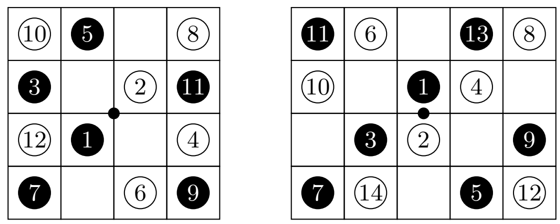

Fig. 4. Example Col games on the 4×4 and 4×5 boards won by White.

3.2 Col on $m \times n$ boards with m and n odd, with $m, n \geq 3$

As a consequence of Theorem 1, only $m \times n$ Col boards with both m and n odd are of interest for solving. Obviously, the second player cannot use the centre strategy, since the first player can at some moment color the centre, to which the second player cannot respond using this strategy. In principle such boards can therefore be either first-player or second-player wins. Only for empty $1 \times n$ boards (see Section 3.3) with odd n we know the solutions, which limits the interesting boards to be solved to empty $m \times n$ boards with both m and n odd and $m, n \geq 3$. Demeur [4] proved that the 3×3, 3×5, 3×7, 3×9, and 5×5 boards are second-player wins (CGT value 0), but his analyses show no general applicable winning strategy for either player on odd $\times$ odd boards.

3.3 Linear Col

We denote Col on a $1 \times n$ board as Linear Col. Of course we already know that Linear Col on even-length boards is a second-player win, so has CGT value 0. The odd-length Linear Col boards are still of interest, since our previous analysis gives no clue. Linear Col is supposedly completely solved, see *Winning Ways* [2], Vol. 1, pp. 49–50. Since there are no proofs given we provide such a proof in Theorem 2. This will be a proof by induction on the length of the chain. To do this we do not only consider just empty, black, or white nodes, but also so-called *tinted* nodes. After coloring a node the Col-condition imposes that its neighbors never may receive the same color. To indicate this we may tint an empty neighbor of a black-colored node white, to indicate that such a node may only be colored White in the future. Similar for a black tint for an empty neighbor of a white node. If an empty node receives both a black and a white tint, it means that this node may not be colored anymore at all.

We use the following notation for this: **B** for a black-colored square, **W** for a white-colored square, **b** for a black-tinted square, **w** for a white-tinted square, **x** for an empty square that can no longer be colored by either player, and **o** for an empty square that still can be be colored by either color. For brevity we omit all edges.

Theorem 2. *Empty Linear Col chains of length n have CGT value 0 for $n > 1$.*

Proof. Note that as soon as a node is white-colored or black-colored (and the neighbors have been updated), it may be removed from the chain, which accordingly splits. It splits also when an empty node can not be colored by any player, since this node may also be removed. Therefore the CGT value for a Linear Col chain can be determined by the values of shorter subchains, in which only end nodes are possibly tinted. For the chains $\mathbf{o}\cdots\mathbf{o}$, $\mathbf{b}\cdots\mathbf{b}$, and $\mathbf{w}\cdots\mathbf{w}$ we consider only options in the left half of the chain, for other chains we consider all options. Of course symmetric chains like $\mathbf{b}\cdots\mathbf{o}$, and $\mathbf{o}\cdots\mathbf{b}$ have the same values, whereas swapping **b**s and **w**s in a chain yields the negation of the CGT value. **0** denotes the Linear Col chain of zero length, of course having value 0.

The analyses below always proceed in (at most) five steps: 1) determine the options of the chain; 2) simplify the options by removing colored and uncolorable nodes; 3) replace the options by their CGT values; 4) remove dominated options; and 5) determine the CGT value of the original chain.

Base cases: $1 \times k$ chains with $k \leq 4$ have the following values:

$k = 1$**:** $\mathbf{o} = \{\mathbf{B}|\mathbf{W}\} = \{\mathbf{0}|\mathbf{0}\} = \{0|0\} = *$; $\mathbf{b} = \{\mathbf{B}|\} = \{\mathbf{0}|\} = \{0|\} = 1$; similarly $\mathbf{w} = -1$.

$k = 2$**:** $\mathbf{oo} = \{\mathbf{Bw}|\mathbf{Wb}\} = \{\mathbf{w}|\mathbf{b}\} = \{-1|1\} = 0$; $\mathbf{bo} = \{\mathbf{Bw}, \mathbf{xB}|\mathbf{bW}\} = \{\mathbf{w}, \mathbf{0}|\mathbf{b}\} = \{-1, 0|1\} = \{0|1\} = 1/2$; similarly $\mathbf{ob} = 1/2$, $\mathbf{wo} = \mathbf{ow} = -1/2$; $\mathbf{bb} = \{\mathbf{Bx}|\} = \{\mathbf{0}|\} = \{0|\} = 1$; similarly $\mathbf{ww} = -1$; $\mathbf{bw} = \{\mathbf{Bw}|\mathbf{bW}\} = \{\mathbf{w}|\mathbf{b}\} = \{-1|1\} = 0$; similarly $\mathbf{wb} = 0$.

$k = 3$**:** $\mathbf{ooo} = \{\mathbf{Bwo}, \mathbf{wBw}|\mathbf{Wbo}, \mathbf{bWb}\} = \{\mathbf{wo}, \mathbf{w}+\mathbf{w}|\mathbf{bo}, \mathbf{b}+\mathbf{b}\} = \{-1/2, -2|1/2, 2\} = \{-1/2|1/2\} = 0$; $\mathbf{boo} = \{\mathbf{Bwo}, \mathbf{xBw}, \mathbf{bwB}|\mathbf{bWb}, \mathbf{bbW}\} = \{\mathbf{wo}, \mathbf{w}, \mathbf{bw}|\mathbf{b} + \mathbf{b}, \mathbf{bb}\} = \{-1/2, -1, 0|2, 1\} = \{0|1\} = 1/2$; similarly $\mathbf{oob} = 1/2$, $\mathbf{woo} = \mathbf{oow} = -1/2$; $\mathbf{bob} = \{\mathbf{Bwb}, \mathbf{xBx}|\mathbf{bWb}\} = \{\mathbf{wb}, \mathbf{0}|\mathbf{b} + \mathbf{b}\} = \{0, 0|2\} = \{0|2\} = 1$; similarly $\mathbf{wow} = -1$; $\mathbf{bow} = \{\mathbf{Bww}, \mathbf{xBw}|\mathbf{bWx}, \mathbf{bbW}\} = \{\mathbf{ww}, \mathbf{w}|\mathbf{b}, \mathbf{bb}\} = \{-1, -1|1, 1\} = \{-1|1\} = 0$; similarly $\mathbf{wob} = 0$.

$k = 4$**:** $\mathbf{oooo} = \{\mathbf{Bwoo}, \mathbf{wBwo}|\mathbf{Wboo}, \mathbf{bWbo}\} = \{\mathbf{woo}, \mathbf{w} + \mathbf{wo}|\mathbf{boo}, \mathbf{b} + \mathbf{bo}\} = \{-1/2, -1 1/2|1/2, 1 1/2\} = \{-1/2|1/2\} = 0$; $\mathbf{booo} = \{\mathbf{Bwoo}, \mathbf{xBwo}, \mathbf{bwBw}, \mathbf{bowB}|\ \mathbf{bWbo}, \mathbf{bbWb}, \mathbf{bobW}\} = \{\mathbf{woo}, \mathbf{wo}, \mathbf{bw} + \mathbf{w}, \mathbf{bow}|\mathbf{b} + \mathbf{bo}, \mathbf{bb} + \mathbf{b}, \mathbf{bob}\} = \{-1/2, -1/2, -1, 0|1 1/2, 2, 1\} = \{0|1\} = 1/2$; similarly $\mathbf{ooob} = 1/2$, $\mathbf{wooo} = \mathbf{ooow} = -1/2$; $\mathbf{boob} = \{\mathbf{Bwob}, \mathbf{xBwb}|\mathbf{bWbb}\} = \{\mathbf{wob}, \mathbf{wb}|\mathbf{b} + \mathbf{bb}\} = \{0, 0|2\} = \{0|2\} = 1$; similarly $\mathbf{woow} = -1$; $\mathbf{boow} = \{\mathbf{Bwow}, \mathbf{xBww}, \mathbf{bwBw}|\mathbf{bWbw}, \mathbf{bbWx}, \mathbf{bobW}\} = \{\mathbf{wow}, \mathbf{ww}, \mathbf{bw}+\mathbf{w}|\mathbf{b}+\mathbf{bw}, \mathbf{bb}, \mathbf{bob}\} = \{-1, -1, -1|1, 1, 1\} = \{-1|1\} = 0$; similarly $\mathbf{woob} = 0$.

So for $1 \le k \le 4$ we have

$$\begin{aligned}&\mathbf{o} = *, \mathbf{b} = 1, \mathbf{w} = -1, \mathbf{o}\cdots\mathbf{o} = 0,\\ &\mathbf{b}\cdots\mathbf{o} = \mathbf{o}\cdots\mathbf{b} = 1/2, \mathbf{w}\cdots\mathbf{o} = \mathbf{o}\cdots\mathbf{w} = -1/2,\\ &\mathbf{b}\cdots\mathbf{b} = 1, \mathbf{w}\cdots\mathbf{w} = -1, \mathbf{b}\cdots\mathbf{w} = \mathbf{w}\cdots\mathbf{b} = 0\end{aligned} \tag{1}$$

Induction hypothesis: suppose Eq. (1) holds for chains of length up to $k-1$.
Induction steps: consider a chain of length $k \ge 5$. We then have the following subcases, where a '$\cdots$' now indicates a sequence of nodes **o**, not of arbitrary length, but the length needed to have a complete chain of length k. For entries with chains '$\cdots$' at both sides of the colored square a range of possible entries is meant such that all combinations of left and right lengths are included with always a total length of k.

$\mathbf{o}\cdots\mathbf{o}$: B moves gives $\{\mathbf{Bw}\cdots\mathbf{o}, \mathbf{wBw}\cdots\mathbf{o}, \mathbf{o}\cdots\mathbf{wBw}\cdots\mathbf{o}\} = \{\mathbf{w}\cdots\mathbf{o}, \mathbf{w} + \mathbf{w}\cdots\mathbf{o}, \mathbf{o}\cdots\mathbf{w} + \mathbf{w}\cdots\mathbf{o}\} = \{-1/2, -1 1/2, -1\} = \{-1/2\}$. Similarly, W moves gives $\{1/2\}$. So $\mathbf{o}\cdots\mathbf{o} = \{-1/2|1/2\} = 0$.

$\mathbf{b}\cdots\mathbf{o}$: B moves gives $\{\mathbf{Bw}\cdots\mathbf{o}, \mathbf{xBw}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{wBw}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{wBw}, \mathbf{b}\cdots\mathbf{wB}\}$ $= \{\mathbf{w}\cdots\mathbf{o}, \mathbf{w}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{w}+\mathbf{w}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{w}+\mathbf{w}, \mathbf{b}\cdots\mathbf{w}\} = \{-1/2, -1/2, -1/2, -1, 0\} = \{0\}$. W moves gives $\{\mathbf{bWb}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{bWb}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{bWb}, \mathbf{b}\cdots\mathbf{bW}\}$ $= \{\mathbf{b}+\mathbf{b}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{b}+\mathbf{b}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{b}+\mathbf{b}, \mathbf{b}\cdots\mathbf{b}\} = \{1 1/2, 1 1/2, 2, 1\} = \{1\}$. So $\mathbf{b}\cdots\mathbf{o} = \{0|1\} = 1/2$. Similarly $\mathbf{o}\cdots\mathbf{b} = 1/2$, $\mathbf{w}\cdots\mathbf{o} = \mathbf{o}\cdots\mathbf{w} = -1/2$.

$\mathbf{b}\cdots\mathbf{b}$: B moves gives $\{\mathbf{Bw}\cdots\mathbf{b}, \mathbf{xBw}\cdots\mathbf{b}, \mathbf{b}\cdots\mathbf{wBw}\cdots\mathbf{b}\} = \{\mathbf{w}\cdots\mathbf{b}, \mathbf{w}\cdots\mathbf{b}, \mathbf{b}\cdots\mathbf{w}+\mathbf{w}\cdots\mathbf{b}\} = \{0, 0, 0\} = \{0\}$. W moves gives $\{\mathbf{bWb}\cdots\mathbf{b}, \mathbf{b}\cdots\mathbf{bWb}\cdots\mathbf{b}\}$ $= \{\mathbf{b}+\mathbf{b}\cdots\mathbf{b}, \mathbf{b}\cdots\mathbf{b}+\mathbf{b}\cdots\mathbf{b}\} = \{2, 2\} = \{2\}$. So $\mathbf{b}\cdots\mathbf{b} = \{0|2\} = 1$. Similarly $\mathbf{w}\cdots\mathbf{w} = -1$.

$\mathbf{b}\cdots\mathbf{w}$: B moves gives $\{\mathbf{Bw}\cdots\mathbf{w}, \mathbf{xBw}\cdots\mathbf{w}, \mathbf{b}\cdots\mathbf{wBw}\cdots\mathbf{w}, \mathbf{b}\cdots\mathbf{wBw}\} = \{\mathbf{w}\cdots\mathbf{w}, \mathbf{w}\cdots\mathbf{w}, \mathbf{b}\cdots\mathbf{w}+\mathbf{w}\cdots\mathbf{w}, \mathbf{b}\cdots\mathbf{w}+\mathbf{w}\} = \{-1, -1, -1, -1\} = \{-1\}$. Similarly W moves gives $\{1\}$. So $\mathbf{b}\cdots\mathbf{w} = \{-1|1\} = 0$. Similarly $\mathbf{w}\cdots\mathbf{b} = 0$.

This means that based on the assumption that Eq. (1) holds for chain length $k-1$ it follows that it holds for chain length k. Combined with the base cases, Eq. (1) consequently holds for arbitrary length chains. □

Concludingly, all empty $1 \times n$ Col boards are second-player wins (CGT value 0), except the 1×1 board has value $*$, and so is a trivial first-player win.

4 Partisan Snort on Rectangular Boards

Although standard Col and Snort are very similar games, it turns out that they differ considerably in CGT outcomes and values. In this section we further focus on the standard (partisan) version of Snort.

4.1 Snort on $m \times n$ boards with m and n even

For $m \times n$ Snort boards with m and n both even we found that the second player always can win. This can easily be proven as stated in Theorem 3.

Theorem 3. *All empty $m \times n$ Snort boards with m and n even are second-player wins and thus have CGT value 0.*

Proof. White as second player follows the centre strategy. So after every black move, White maintains opposite colored squares wrt the centre of the board, meaning that White necessarily makes the last move and wins. □

Note that this strategy is exactly the same as used in Col on boards with at least one dimension even. Although the Snort-condition differs, for even $\times$ even boards the symmetry applied makes sure that after any black move obeying the Snort-condition the white response automatically also obeys this condition.

An example Snort game on the 4×4 board is shown in Fig. 5. The first eight moves are the same as the Col game shown in Fig. 4. This is possible since for these moves it holds that there are no colored neighbors yet. From the ninth move on every move necessarily has a colored neighbor and therefore the Snort game now differs from the Col game.

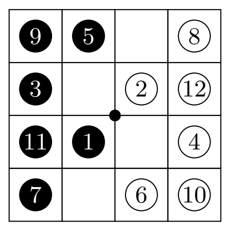

Fig. 5. Example Snort game on the 4×4 board won by White.

4.2 Snort on $m \times n$ boards with m and/or n odd

For $m \times n$ Snort boards with m and/or n odd the second player cannot use the above copy-strategy to win the game. Instead, we found that the first player always can win, see Theorem 4.

Theorem 4. *All empty $m \times n$ Snort boards with m and/or n odd are first-player wins.*

Proof. First assume that both m and n are odd. Black as first player then starts coloring the single centre square and then follows the centre strategy. Although the black centre inhibits its neighbors to be colored white in the future, it does not hamper Black, so the centre strategy still is always possible. Therefore, Black can maintain opposite colored squares wrt the centre after every white move (of course excluding the centre). Therefore Black makes the last move and wins.

When only one of m and n is odd (arbitrarily suppose m), then the number of rows is odd and the centre of the board is in the middle row between the two middle squares. Now Black as first player colors one of these two middle squares and then again can use the centre strategy. Of course White cannot use the second middle square. So the centre strategy again guarantees Black to make the last move and win. □

The latter result, stating that the first player wins on a board with at least one dimension odd, does not give the CGT value of these Snort boards, which in principle can be any fuzzy value (like a fair switch or a nimber). Example games where the first player uses this winning strategy are given in Fig. 6.

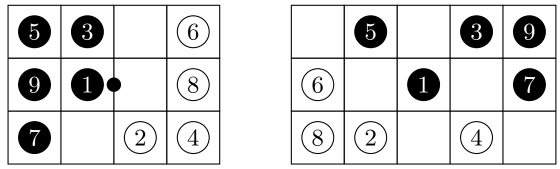

Fig. 6. Example Snort games on the 3×4 and 3×5 boards won by Black.

As a consequence of Theorems 3 and 4 strategically solving rectangular Snort boards is of no more interest, since the dimensions of the board fully determine the winner. Regarding full (CGT) values, determining values of rectangular Snort boards with at least one dimension odd is still of interest.

4.3 Linear Snort

Since $1 \times n$ chains are instances of odd $\times$ even or odd $\times$ odd boards, and since both these board categories are first-player wins for Snort, we know that all Linear Snort boards are first-player wins with fuzzy values, like nimbers or fair switches. To see if we can find some pattern we determined many CGT values for Linear Snort, using the CGSUITE system [7]. The following values where obtained for various lengths n of the board: $n = 1$: $*$; $n = 2$: ± 1; $n = 3$: ± 2; $n = 4$: $\pm\{2|1\}$; $n = 5$: $\pm(1,\{3|0\})$; $n = 6$: $*$; $n = 7$: $\pm(1,\{4|3||*|-1*\},\{4|3||\pm 1,\{1*|*\}\})$; $n = 8$: $\pm\{\{5|2\},\{5|2*\}|\pm 2,\{2|1||0|-1\},\{2*|-2\}\}$; and $n = 9$: $\pm(2*)$. For lengths 10 to 12 we found fair switches with canonical forms consisting of 273, 628, and 1954 symbols respectively, which we do not reproduce here. Unsurprisely we did not find any pattern in these CGT values.

5 Conclusions and Future Research

We summarize our main results in Table 1. In this table, for every board type we give the outcome class.'?' indicates that the outcome in general is unknown. Instances of outcome class $\mathcal{P}$ have CGT value 0, while instances of outcome class $\mathcal{N}$ have fuzzy CGT values (nimbers or fair switches). For Linear Col and Snort we summarize the results in Table 2.

For all types tabulated all instances belong to the same type, except that for odd $\times$ odd Col we do know that some instances are second-player wins, but do not know if first-player wins also occur.

Game	even × even	odd × even	odd × odd
Col	$\mathcal{P}$	$\mathcal{P}$	?
Snort	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$

Table 1. Outcome classes for Col and Snort on boards of different types.

Game	even n	odd n
Linear Col	$\mathcal{P}$	$\mathcal{P}$
Linear Snort	$\mathcal{N}$	$\mathcal{N}$

Table 2. Outcome classes for Linear Col and Snort on chains of length $n > 1$.

All results in [3, 2, 4] fully support our results. Also, all values in this paper were checked with the CGSUITE system [7], and no discrepancies were found.[1]

For future research we will focus on finding optimal strategies for odd × odd Col boards with both dimensions ≥ 3. We are also interested in results for Col and Snort played on other graphs than rectangular boards. Moreover, we are interested in other bicoloring games. For impartial versions of Col and Snort (dubbed iCol and iSnort) played on rectangular boards we already performed such an analysis [8].

Acknowledgement

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[1] We greatfully used the CGSUITE code for Snort on a rectangular board, as provided in Svenja Huntemann's Ph.D. thesis [5]. The CGSUITE code for the board implementation of Col was a simple adaptation thereof.