Solving Bicoloring-Graph Games on Rectangular Boards – Part 2: impartial Col and Snort

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Abstract. As a sequel to an investigation of the standard (*partisan*) versions of Col and Snort on rectangular boards, we defined *impartial* versions of both games (dubbed *iCol* and *iSnort*). These have the same coloring conditions as their partisan versions, but either player is allowed to use at any move a black or a white stone. For these two games similar strategies show that with both dimensions odd the first player can win, otherwise it is a second-player win. For both Linear versions, analyses using Combinatorial Game Theory show that the even-length chains have value 0, the odd-length chains value $*$.

1 Introduction

In a previous paper [8] we analyzed two well-known bicoloring-graph games on rectangular boards, namely Col and Snort. They were the standard (partisan) versions of these combinatorial games. Both are similar in the sense that both players alternately color a node in the graph, where one player may only color it black, the other only white. The two games differ in their conditions for coloring: in Col neighboring nodes may not be colored the same (the Col-condition), while in Snort they may not be colored differently (the Snort-condition). These two games then were largely solved.

In the present paper we introduce impartial versions of both games, denoted as iCol and iSnort, focussing on one type of graphs, namely rectangular boards. The games are played with the same restrictions on coloring neighboring nodes as their partisan versions (the Col- and Snort-conditions), but differ in the property that both players always may use either color. This on one hand makes playing them easier, since values of games belong to just two outcome classes (see Section 2), but on the other hand makes them more complex, since for neither player it is possible to build significant advantages due to the nature of the games.

Since these games are new to our knowledge, there is no previous scientific literature on them. We only found a single mention of iCol, under the name Bichrome [7], though it was just presented as a fun game and not analyzed mathematically, notably not in the framework of the Combinatorial Game Theory. For iSnort we found no mention in the literature at all.

2 Combinatorial Game Theory for iCol and iSnort

In this section we give a short introduction to the Combinatorial Game Theory (CGT in short) as far as relevant for the games discussed in this paper. For a more thorough introduction, we refer to the literature, in particular [4, 2, 1].

In a combinatorial game, the players are conventionally called Left and Right. Left starts the game. A game G is then represented by its left and right *options* G^L and G^R, so $G = \{G^L \mid G^R\}$. In this representation, G^L and G^R stand for sets of games that players Left and Right, respectively, can reach by making one move in the game. The *value* of a game indicates how good a game is for a player, where positive values indicate an advantage for Left and negative values an advantage for Right. Then there are four possible outcome classes.

1. The class $\mathcal{L}$ consists of all positions where Left wins, irrespective of who moves first. These positions have strictly positive values.
2. The class $\mathcal{R}$ consists of all positions where Right wins, irrespective of who moves first. These positions have strictly negative values.
3. The class $\mathcal{N}$ consists of all positions where the player to move (the next player) wins. These positions have fuzzy values (incomparable with 0).
4. The class $\mathcal{P}$ consists of all positions where the player to move loses, so the previous player wins. These positions all have value 0.

For impartial games, like iCol and iSnort, it holds that they can only take *nimbers* as values and hence that all positions have outcome class $\mathcal{N}$ or $\mathcal{P}$.

2.1 Nimbers

The simplest nimber game is the endgame $\{|\}$, denoted as $*0$. In this position, no player has any available moves, so it is a loss for the player to move and hence a second-player win. Its outcome class is therefore $\mathcal{P}$. Note that this game is the only game being both a nimber and a number, hence $*0 = 0$.

Besides the endgame $*0$, the most important simple game is the one denoted as $*1$, often notated as just $*$. It is defined as $* = \{0 \mid 0\}$, where the player to move has just one option, leading to the endgame. Therefore, whereas 0 is a game where the second player to move wins (since trivially the next player cannot move), $*$ is a game where the first player to move wins. A trivial example in both iCol and iSnort is a lone empty square. Nimbers take their name from the values that can occur in the Nim game [3], where each player has the same options. They are formally defined as $*n = \{*0, *1, *2, \cdots, *(n-1) \mid *0, *1, *2, \cdots, *(n-1)\}$. In case that not all options for a player are consecutive nimbers starting from $*0$, it follows from CGT that the Mex() function applied to the options gives the nimber value of the parent game. The Mex() function (*Minimal excludant*) is the lowest non-negative integer **not** in a set of integers. In case of sums of nimbers they are added pairwise using the Nim-addition rule, which effectively boils down to exclusive-oring the binary representations of the nimbers.

All nimbers other than $*0$ are fuzzy (incomparable with 0) and denote first-player wins. Their outcome class is therefore $\mathcal{N}$.

3 Impartial Col and Snort

We noted in previous research [8] that most Col and Snort games on rectangular boards (except Col on odd × odd boards) have known outcomes and easy strategies guaranteeing these outcomes. We then were interested to see if such winning strategies are also possible for impartial versions of Col and Snort. These are defined as follows.

Definition 1. *Impartial Col (iCol in short) and impartial Snort (iSnort) are coloring games on graphs, where the same restrictions on possible colors of neighboring nodes apply as in Col and Snort, respectively, but both players are free to use any of the two colors (Black or White) on their turn.*

By this definition both players have exactly the same possible moves in any game position, and so are truly impartial games. We therefore further do not use Black and White for the names of the players in iCol and iSnort, but Left (first to move) and Right. As stated in Section 2 all impartial games, including iCol and iSnort, have only nimbers as possible values.

3.1 iCol on rectangular boards

For $m \times n$ iCol boards with m and/or n even the second player always can win. This is proven in Theorem 1.

Theorem 1. *All empty $m \times n$ iCol boards with m and/or n even are second-player wins and thus have CGT value 0.*

Proof. The second player can use a centre strategy similar as in Col, i.e. the second player always moves symmetric wrt to the centre of the board using the opposite color as the previous move. Therefore, after every second-player's move the board is centre-symmetric with opposite colors. Consequently, the second player makes the last move and wins. □

Since in iCol (and later iSnort) both players can use both colors, we add the term "same" or "opp" to the strategy name, so the winning strategy described in the above theorem is called the *centre-opp strategy*. Of course when the first player just sticks to using one color, we have a standard Col game won by the second player. Example iCol games on the 4 × 4 and 4 × 5 boards where the second player (Right) uses this winning strategy are shown in Fig. 1.

The left diagram shows an iCol game on an even × even board, the right diagram on an even × odd board. Right has chosen to always use the centre-opp strategy, guaranteeing the win. Note that for iCol on an even × even board an alternative winning strategy for the second player would be to use the centre-same strategy. For odd × even and even × odd boards this strategy is not possible, since it might violate the Col-condition.

For $m \times n$ iCol boards with m and n odd the first player always can win. This is proven in Theorem 2.

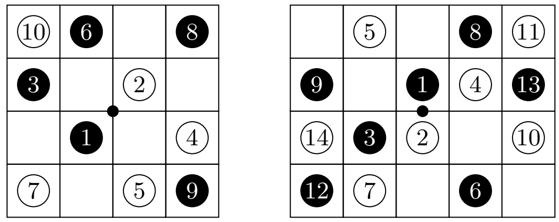

Fig. 1. Example iCol games on the 4×4 and 4×5 boards won by Right.

Theorem 2. *All empty $m \times n$ iCol boards with m and n odd are first-player wins.*

Proof. Contrary to Col, in iCol the first player can easily win by first coloring the centre square arbitrarily (say, black), followed by using the centre-same strategy. This guarantees the first player to make the last move and win. □

An example game where the first player uses this strategy to win the game is given in Fig. 2.

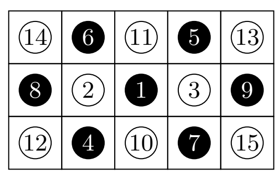

Fig. 2. Example iCol game on the 3×5 board won by Left.

Like in Col [8] we only know that odd $\times$ odd boards have fuzzy values, but since all values in iCol must be nimbers, we know that the values of odd $\times$ odd boards have nimber values $*n$ with $n > 0$.

3.2 Linear iCol

In the following we analyze Linear iCol in a similar way as we did for Linear Col [8]. The only difference is that both players may use both colors (as long as they respect the Col-condition), which makes the analysis longer. On the other hand it suffices to only consider the options of one player, since the other player has exactly the same options by the impartial nature of the game; this shortens the analysis.

After coloring a node the Col-condition imposes that its neighbors never may receive the same color. To indicate this we may tint an empty neighbor of a black-colored node white, to show that such a node may only be colored White in the future. Similar for a black tint for an empty neighbor of a white node. If an empty node receives both a black and a white tint, it means that this node may not be colored anymore at all.

We use the following notation for this: **B** for a black-colored square, **W** for a white-colored square, **b** for a black-tinted square, **w** for a white-tinted square, **x** for an empty square that can no longer be colored by either player, and **o** for an empty square that still can be be colored by either color. For brevity we omit all edges. Our result is stated in Theorem 3.

Theorem 3. *Empty Linear iCol chains have CGT value 0 for even length and ∗ for odd length.*

Proof. Again when a node is colored (and the neighbors have been updated), it may be removed from the graph, which accordingly splits. It splits also when an empty node can not be colored, since this node may also be removed. Therefore the CGT value for a Linear iCol chain can be determined by the values of shorter subchains, in which only end nodes are possibly tinted. For the chains $\mathbf{o}\cdots\mathbf{o}$, $\mathbf{b}\cdots\mathbf{b}$, and $\mathbf{w}\cdots\mathbf{w}$ we consider only options in the left half of the chain, for other chains we consider all options. Of course symmetric chains like $\mathbf{b}\cdots\mathbf{o}$, and $\mathbf{o}\cdots\mathbf{b}$ have the same values, just as swapping **b**s and **w**s in a chain (yielding the negation of the CGT value, which for nimbers has no effect). **0** denotes the Linear iCol chain of zero length, of course having value 0.

The analyses below always proceed in (at most) five steps: 1) determine the options of the chain; 2) simplify the options by removing colored and uncolorable nodes; 3) replace the options by their CGT values; 4) remove dominated options; and 5) determine the CGT value of the original chain.

Base cases: $1 \times k$ chains with $k \leq 4$ have the following values:

$k = 1$**:** $\mathbf{o} = \{\mathbf{B}, \mathbf{W}\} = \{\mathbf{0}, \mathbf{0}\} = \{0, 0\} = \{0\} = *$; $\mathbf{b} = \{\mathbf{B}\} = \{\mathbf{0}\} = \{0\} = *$; similarly $\mathbf{w} = *$.

$k = 2$**:** $\mathbf{oo} = \{\mathbf{Bw}, \mathbf{Wb}\} = \{\mathbf{w}, \mathbf{b}\} = \{*, *\} = \{*\} = 0$; $\mathbf{bo} = \{\mathbf{Bw}, \mathbf{xB}, \mathbf{bW}\} = \{\mathbf{w}, \mathbf{0}, \mathbf{b}\} = \{*, 0, *\} = \{0, *\} = *2$; similarly, $\mathbf{ob} = \mathbf{wo} = \mathbf{ow} = *2$; $\mathbf{bb} = \{\mathbf{Bx}\} = \{\mathbf{0}\} = \{0\} = *$; similarly $\mathbf{ww} = *$; $\mathbf{bw} = \{\mathbf{Bw}, \mathbf{bW}\} = \{\mathbf{w}, \mathbf{b}\} = \{*, *\} = \{*\} = 0$; similarly $\mathbf{wb} = 0$.

$k = 3$**:** $\mathbf{ooo} = \{\mathbf{Bwo}, \mathbf{wBw}, \mathbf{Wbo}, \mathbf{bWb}\} = \{\mathbf{wo}, \mathbf{w}{+}\mathbf{w}, \mathbf{bo}, \mathbf{b}{+}\mathbf{b}\} = \{*2, 0, *2, 0\} = \{0, *2\} = *$; $\mathbf{boo} = \{\mathbf{Bwo}, \mathbf{xBw}, \mathbf{bwB}, \mathbf{bWb}, \mathbf{bbW}\} = \{\mathbf{wo}, \mathbf{w}, \mathbf{bw}, \mathbf{b}{+}\mathbf{b}, \mathbf{bb}\} = \{*2, *, 0, 0, *\} = \{0, *, *2\} = *3$; similarly $\mathbf{oob} = \mathbf{woo} = \mathbf{oow} = *3$; $\mathbf{bob} = \{\mathbf{Bwb}, \mathbf{xBx}, \mathbf{bWb}\} = \{\mathbf{wb}, \mathbf{0}, \mathbf{b} + \mathbf{b}\} = \{0, 0, 0\} = \{0\} = *$; similarly $\mathbf{wow} = *$; $\mathbf{bow} = \{\mathbf{Bww}, \mathbf{xBw}, \mathbf{bWx}, \mathbf{bbW}\} = \{\mathbf{ww}, \mathbf{w}, \mathbf{b}, \mathbf{bb}\} = \{*, *, *, *\} = \{*\} = 0$; similarly $\mathbf{wob} = 0$.

$k = 4$**:** $\mathbf{oooo} = \{\mathbf{Bwoo}, \mathbf{wBwo}, \mathbf{Wboo}, \mathbf{bWbo}\} = \{\mathbf{woo}, \mathbf{w} + \mathbf{wo}, \mathbf{boo}, \mathbf{b} + \mathbf{bo}\} = \{*3, *3, *3, *3\} = \{*3\} = 0$; $\mathbf{booo} = \{\mathbf{Bwoo}, \mathbf{xBwo}, \mathbf{bwBw}, \mathbf{bowB}, \mathbf{bWbo}, \mathbf{bbWb}, \mathbf{bobW}\} = \{\mathbf{woo}, \mathbf{wo}, \mathbf{bw}{+}\mathbf{w}, \mathbf{bow}, \mathbf{b}{+}\mathbf{bo}, \mathbf{bb}{+}\mathbf{b}, \mathbf{bob}\} = \{*3, *2, *, 0, *3, 0, *\} = \{0, *, *2, *3\} = *4$; similarly $\mathbf{ooob} = \mathbf{wooo} = \mathbf{ooow} = *4$; $\mathbf{boob} = \{\mathbf{Bwob}, \mathbf{xBwb}, \mathbf{bWbb}\} = \{\mathbf{wob}, \mathbf{wb}, \mathbf{b} + \mathbf{bb}\} = \{0, 0, 0\} = \{0\} = *$; similarly $\mathbf{woow} = *$; $\mathbf{boow} = \{\mathbf{Bwow}, \mathbf{xBww}, \mathbf{bwBw}, \mathbf{bWbw}, \mathbf{bbWx}, \mathbf{bobW}\} = \{\mathbf{wow}, \mathbf{ww}, \mathbf{bw}{+}\mathbf{w}, \mathbf{b}{+}\mathbf{bw}, \mathbf{bb}, \mathbf{bob}\} = \{*, *, *, *, *, *\} = \{*\} = 0$; similarly $\mathbf{woob} = 0$.

So for $1 \le k \le 4$ we have

$$\begin{aligned}
\mathbf{o} = \mathbf{b} = \mathbf{w} &= * \\
\mathbf{o}\cdots\mathbf{o} &= \begin{cases} 0 & \text{if } k \text{ is even} \\ * & \text{if } k \text{ is odd} \end{cases} \\
\mathbf{b}\cdots\mathbf{o} = \mathbf{w}\cdots\mathbf{o} = \mathbf{o}\cdots\mathbf{b} = \mathbf{o}\cdots\mathbf{w} &= *k \\
\mathbf{b}\cdots\mathbf{b} = \mathbf{w}\cdots\mathbf{w} &= * \\
\mathbf{b}\cdots\mathbf{w} = \mathbf{w}\cdots\mathbf{b} &= 0
\end{aligned} \tag{1}$$

Induction hypothesis: suppose Eq. (1) holds for chains of length up to $k-1$. **Induction steps**: consider a chain of length $k \ge 5$. We then have the following subcases, where a '$\cdots$' now indicates a sequence of nodes **o**, not of arbitrary length, but the length needed to have a complete chain of length k. For entries with chains '$\cdots$' at both sides of the colored square a range of possible entries is meant such that all combinations of left and right lengths are included with always a total length of k.

$\mathbf{o}\cdots\mathbf{o} = \{\mathbf{Bw}\cdots\mathbf{o}, \mathbf{wBw}\cdots\mathbf{o}, \mathbf{o}\cdots\mathbf{wBw}\cdots\mathbf{o}\} = \{\mathbf{w}\cdots\mathbf{o}, \mathbf{w}+\mathbf{w}\cdots\mathbf{o}, \mathbf{o}\cdots\mathbf{w}+\mathbf{w}\cdots\mathbf{o}\} = \{*(k-1), *+*(k-2), \ldots, *(k-2)+*, *(k-1)\}$ (and similar for the first player using **W**, with the same values). For even k we see that every option is either an odd nimber or the Nim-sum of an odd plus even nimber, which is an odd nimber. Therefore the value of $\mathbf{o}\cdots\mathbf{o}$ for even k is 0 (being the $Mex()$ of all-odd nimbers). For odd k we see that every option is either an even nimber or the Nim-sum of two odd nimbers (which is an even nimber). This includes 0, namely when the middle node is colored (black or white), since both subchains are equal then. Therefore the value of $\mathbf{o}\cdots\mathbf{o}$ for odd k is $*$ (being the $Mex()$ of all-even nimbers including 0).

$\mathbf{b}\cdots\mathbf{o} = \{\mathbf{Bw}\cdots\mathbf{o}, \mathbf{xBw}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{wBw}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{wBw}, \mathbf{b}\cdots\mathbf{wB}, \mathbf{bWb}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{bWb}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{bWb}, \mathbf{b}\cdots\mathbf{bW}\} = \{\mathbf{w}\cdots\mathbf{o}, \mathbf{w}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{w}+\mathbf{w}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{w}+\mathbf{w}, \mathbf{b}\cdots\mathbf{w}, \mathbf{b}+\mathbf{b}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{b}+\mathbf{b}\cdots\mathbf{o}, \mathbf{b}\cdots\mathbf{b}+\mathbf{b}, \mathbf{b}\cdots\mathbf{b}\} = \{*(k-1), *(k-2), 0+*(k-3), \ldots, 0+*, 0+0, *+*(k-2), *+*(k-3), \ldots, *+*, *\} = \{0, \ldots, *(k-1)\} = *k$. Similarly $\mathbf{o}\cdots\mathbf{b} = \mathbf{w}\cdots\mathbf{o} = \mathbf{o}\cdots\mathbf{w} = *k$.

$\mathbf{b}\cdots\mathbf{b} = \{\mathbf{Bw}\cdots\mathbf{b}, \mathbf{xBw}\cdots\mathbf{b}, \mathbf{b}\cdots\mathbf{wBw}\cdots\mathbf{b}, \mathbf{bWb}\cdots\mathbf{b}, \mathbf{b}\cdots\mathbf{bWb}\cdots\mathbf{b}\} = \{\mathbf{w}\cdots\mathbf{b}, \mathbf{w}\cdots\mathbf{b}, \mathbf{b}\cdots\mathbf{w}+\mathbf{w}\cdots\mathbf{b}, \mathbf{b}+\mathbf{b}\cdots\mathbf{b}, \mathbf{b}\cdots\mathbf{b}+\mathbf{b}\cdots\mathbf{b}\} = \{0, 0, \ldots, 0, *+*, \ldots, *+*\} = \{0\} = *$. Similarly $\mathbf{w}\cdots\mathbf{w} = *$.

$\mathbf{b}\cdots\mathbf{w} = \{\mathbf{Bw}\cdots\mathbf{w}, \mathbf{xBw}\cdots\mathbf{w}, \mathbf{b}\cdots\mathbf{wBw}\cdots\mathbf{w}, \mathbf{b}\cdots\mathbf{wBw}, \mathbf{bWb}\cdots\mathbf{w}, \mathbf{b}\cdots\mathbf{bWb}\cdots\mathbf{w}, \mathbf{b}\cdots\mathbf{bWx}, \mathbf{b}\cdots\mathbf{bW}\} = \{\mathbf{w}\cdots\mathbf{w}, \mathbf{w}\cdots\mathbf{w}, \mathbf{b}\cdots\mathbf{w}+\mathbf{w}\cdots\mathbf{w}, \mathbf{b}\cdots\mathbf{w}+\mathbf{w}, \mathbf{b}+\mathbf{b}\cdots\mathbf{w}, \mathbf{b}\cdots\mathbf{b}+\mathbf{b}\cdots\mathbf{w}, \mathbf{b}\cdots\mathbf{b}, \mathbf{b}\cdots\mathbf{b}\} = \{*, *, 0+*, 0+*, *+0, *+0, *, *\} = \{*\}$. So $\mathbf{b}\cdots\mathbf{w} = 0$. Similarly $\mathbf{w}\cdots\mathbf{b} = 0$.

This means that based on the assumption that Eq. (1) holds for chain length up to $k-1$ it follows that it holds for chain length k. Combined with the base cases, Eq. (1) consequently holds for arbitrary chain lengths. □

Concludingly, in Linear iCol even-length empty chains have CGT value 0 (second-player wins) and odd-length empty chains have CGT value $*$ (first-player wins). In the latter case the first-player must color the middle square (either black or white).

3.3 iSnort on rectangular boards

For $m \times n$ iSnort boards with m and/or n even the second player always can win. This is proven in Theorem 4.

Theorem 4. *All empty $m \times n$ iSnort boards with m and/or n even are second-player wins and thus have CGT value 0.*

Proof. The second player can again use a copy-strategy as in iCol, but always using the same color as the previous move (the centre-same strategy). Therefore, after every second-player's move the board is centre-symmetric with same colors. Consequently, the second player makes the last move and wins. □

When the first player just sticks to using one color, the game ends after the full board has been filled with one color, which is after an even number of moves, so the second player wins. Example iSnort games on the 4×4 and 4×5 boards where the second player uses this winning strategy are shown in Fig. 3.

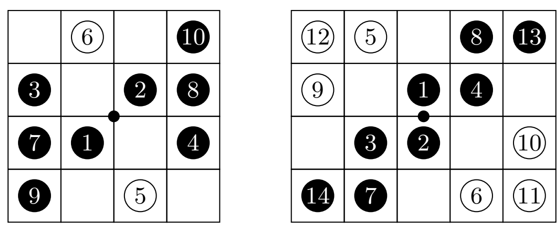

Fig. 3. Example iSnort games on the 4×4 and 4×5 boards won by Right.

The left diagram shows an iSnort game on an even $\times$ even board, the right diagram on an even $\times$ odd board. Right has chosen to always use the centre-same strategy, guaranteeing the win. Note that for iSnort on an even $\times$ even board an alternative winning strategy for the second player would be to use the centre-opp strategy. For odd $\times$ even and even $\times$ odd boards this strategy is not possible, since it might violate the Snort-condition.

Analogously with iCol, for $m \times n$ iSnort boards with m and n odd the first player always can win. This is proven in Theorem 5.

Theorem 5. *All empty $m \times n$ iSnort boards with m and n odd are first-player wins.*

Proof. Like in iCol, the first player can easily win by first coloring the centre square arbitrarily, followed by using the centre-same strategy. Since the number of empty squares after the first move is even, this guarantees the first player to make the last move and win. □

An example game where the first player uses this strategy to win the game is given in Fig. 4.

Again, we only know that such boards have fuzzy values, but since all values in iCol must be nimbers, we know that the values of odd $\times$ odd boards have nimber values $*n$ with $n > 0$.

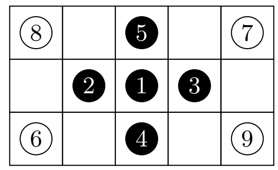

Fig. 4. Example iSnort game on the 3×5 board won by Left.

3.4 Linear iSnort

For Linear iSnort the situation is quite similar as for iCol, namely second-player wins (CGT value 0) for even-length chains, and first-player wins (CGT values $*n$ with $n > 0$) for odd-length chains. For iCol we found that all odd-length chains have value $*$ (Theorem 3). To prove that this is also the case for iSnort we analyze Linear iSnort in a similar way as we did for iCol, using the same notations. Again both players may use both colors, but now the moves must respect the Snort-condition. Our result is stated in Theorem 6.

Theorem 6. *Empty Linear iSnort chains have CGT value 0 for even length and $*$ for odd length.*

Proof. This proof is analogous as the proof of Theorem 3, of course except the Snort-condition on neighboring squares instead of the Col-condition. We just give the main parts of the proof and leave the complete analysis as an exercise.

Base cases: $1 \times k$ chains with $k \le 4$ have the following values:
$k = 1$**:** $\mathbf{o} = \mathbf{b} = \mathbf{w} = \{0\} = *$.
$k = 2$**:** $\mathbf{oo} = \{*\} = 0$; $\mathbf{bo} = \mathbf{ob} = \mathbf{wo} = \mathbf{ow} = \{0, *\} = *2$; $\mathbf{bb} = \mathbf{ww} = \{*\} = 0$; $\mathbf{bw} = \mathbf{wb} = \{0\} = *$.
$k = 3$**:** $\mathbf{ooo} = \{0, *2\} = *$; $\mathbf{boo} = \mathbf{oob} = \mathbf{woo} = \mathbf{oow} = \{0, *, *2\} = *3$; $\mathbf{bob} = \mathbf{wow} = \{0\} = *$; $\mathbf{bow} = \mathbf{wob} = \{*\} = 0$.
$k = 4$**:** $\mathbf{oooo} = \{*3\} = 0$; $\mathbf{booo} = \mathbf{ooob} = \mathbf{wooo} = \mathbf{ooow} = \{0, *, *2, *3\} = *4$; $\mathbf{boob} = \mathbf{woow} = \{*\} = 0$; $\mathbf{boow} = \mathbf{woob} = \{0\} = *$.

So for $1 \le k \le 4$ we have

$$\begin{aligned}
\mathbf{o} = \mathbf{b} = \mathbf{w} &= * \\
\mathbf{o}\cdots\mathbf{o} &= \begin{cases} 0 & \text{if } k \text{ is even} \\ * & \text{if } k \text{ is odd} \end{cases} \\
\mathbf{b}\cdots\mathbf{o} = \mathbf{w}\cdots\mathbf{o} = \mathbf{o}\cdots\mathbf{b} = \mathbf{o}\cdots\mathbf{w} &= *k \\
\mathbf{b}\cdots\mathbf{b} = \mathbf{w}\cdots\mathbf{w} &= \begin{cases} 0 & \text{if } k \text{ is even} \\ * & \text{if } k \text{ is odd} \end{cases} \\
\mathbf{b}\cdots\mathbf{w} = \mathbf{w}\cdots\mathbf{b} &= \begin{cases} * & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}
\end{aligned} \tag{2}$$

Induction hypothesis: suppose Eq. (2) holds for chains of length up to $k - 1$.
Induction steps: consider a chain of length $k \ge 5$. We then have the following subcases.

$\mathbf{o} \cdots \mathbf{o}$ $= \{*(k-1), * + *(k-2), \ldots, *(k-2) + *, *(k-1)\}$ (and similar for the first player using $\mathbf{W}$, with the same values). The value of $\mathbf{o} \cdots \mathbf{o}$ for even k is 0, for odd k it is $*$.

$\mathbf{b} \cdots \mathbf{o}$: We have to distinguish cases where the chain has odd or even length, and whether the player uses Black or White as color.
We first consider even k. Using Black the sums of the left and right subchains are $*(k-1)$, $*(k-1)$, $*(k-3)$, $*(k-3)$, etc, i.e. all odd nimbers from $*$ to $*(k-1)$. Using White the sums of the left and right subchains are $*(k-2)$, $*(k-4)$, $*(k-4)$, etc, i.e. all even nimbers from 0 to $*(k-2)$. Taking all options together we see that all nimbers from 0 to $*(k-1)$ are included, which means that the value of the total chain for even k is $*k$.
For odd k the analysis is quite similar, this time leading to the series $*(k-1)$, $*(k-3)$, $*(k-3)$, $*(k-5)$, $*(k-5)$, etc for the options using Black, including all even nimbers from 0 to $*(k-1)$; for the options using White we obtain the series $*(k-2)$, $*(k-2)$, $*(k-4)$, $*(k-4)$, etc, again including all odd nimbers from $*$ to $*(k-2)$. Taking all options together we see that again all nimbers from 0 to $*(k-1)$ are included, which means that the value of the total chain for odd k is also $*k$.

$\mathbf{b} \cdots \mathbf{b}$: We again differentiate between the Black and White options.
For options using Black the chain is split in two parts, with either value 0 (even-length $\mathbf{b} \cdots \mathbf{b}$ subchains) or $*$ (odd-length $\mathbf{b} \cdots \mathbf{b}$ subchains). For options using White the chain is split in two parts, with either value $*$ (even-length $\mathbf{b} \cdots \mathbf{w}$ subchains) or 0 (odd-length $\mathbf{b} \cdots \mathbf{w}$ subchains). In either case, for even k the sum of the two splits has odd length, with sum value $*$, which means the original chain has value 0; for odd k the sum of the two splits has even length, with sum value 0, which means the original chain has value $*$. Similarly, a chain $\mathbf{w} \cdots \mathbf{w}$ with length k also has value 0 if k is even, and value $*$ if k is odd.

$\mathbf{b} \cdots \mathbf{w}$: We again differentiate between the Black and White options.
For options using Black the chain is split into a $\mathbf{b} \cdots \mathbf{b}$ subchain (odd-length $*$, even-length 0) and a $\mathbf{b} \cdots \mathbf{w}$ subchain (odd-length 0, even-length $*$). For options using White the chain is split into a $\mathbf{b} \cdots \mathbf{w}$ subchain (odd-length 0, even-length $*$) and a $\mathbf{w} \cdots \mathbf{w}$ subchain (odd-length $*$, even-length 0). In either case, for even k the sum of the two splits has sum value 0, which means the original chain has value $*$; for odd k the sum of the two splits has sum value $*$, which means the original chain has value 0. Similarly, a chain $\mathbf{w} \cdots \mathbf{b}$ with length k also has value $*$ if k is even, and value 0 if k is odd.

This means that based on the assumption that Eq. (2) holds for chain length up to $k-1$ it follows that it holds for chain length k. Combined with the base cases, Eq. (2) consequently holds for arbitrary length chains. □

Concludingly, like in Linear iCol, in Linear iSnort even-length empty chains have CGT value 0 (second-player wins) and odd-length empty chains have CGT value $*$ (first-player wins). In the latter case the first-player must color the middle square (either black or white).

4 Conclusions and Future Research

We summarize our main results in Table 1, where we give for all board types the corresponding outcome class. Note that the results for even $\times$ odd boards are equal to their equivalent odd $\times$ even boards obtained by a 90° rotation.

Game	even $\times$ even	odd $\times$ even	odd $\times$ odd
iCol	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$
iSnort	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$

Table 1. Outcome classes for iCol and iSnort on boards of different types.

This table shows that iCol and iSnort on rectangular boards are solved games, though their values on odd $\times$ odd boards can vary (nimbers $*n$ with $n > 0$).

For Linear iCol and iSnort we do have precise CGT values, given in Table 2.

Game	even k	odd k
iCol	0	$*$
iSnort	0	$*$

Table 2. CGT values for Linear iCol and iSnort on chains of length k.

All values in this paper were checked with the CGSUITE [6] system and fully agree with our findings.[1] For future research we are also interested in results for iCol and iSnort played on other graphs than rectangular boards.

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[1] The CGSUITE code for the board implementations of iCol and iSnort were simple adaptations of the ones for Col and Snort as used in [8].